Chapter 3.3 Applications

Problem 1. Compute the following areas:

i) area limited by the following curves $y = x$ and $y = 2 - x^2$;

ii) area of the region $A = \{(x, y) \in \mathbb{R}^2 \;:\; x, y > 0,\; a^2 y \leq x^3 \leq b^2 y,\; p^2 x \leq y^3 \leq q^2 x, \}$, where $0 < a < b$ y $0 < p < q$.

iii) area defined by the curves $xy = 4$, $xy = 8$, $xy^3 = 5$ and $xy^3 = 15$.

Solution: $i)$ $9/2$; $ii)$ $(b - a)(q - p)/2$; $iii)$ $2 \log 3$.

Problem 2. Find the volumes of the regions defined by:

i) $z = x^2 + 3y^2,\; z = 9 - x^2$.

ii) $x^2 + 2y^2 = 2,\; z = 0,\; x + y + 2z = 2$.

Solution: $i)$ $9\pi\sqrt{2}/4$; $ii)$ π.

Problem 3. Compute the following volumes:

i) volume defined by the intersection of the cylinder $x^2 + y^2 \leq 4$ and the ball $x^2 + y^2 + z^2 \leq 16$;

ii) volume of the region bounded by the cones $z = 1 - \sqrt{x^2 + y^2}$ and $z = -1 + \sqrt{x^2 + y^2}$;

iii) volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$ with $z \geq 0$;

iv) volume of the region bounded by $x^2 + y^2 + z^2 \leq 2,\; x^2 + y^2 \leq z$ and $z \leq 6/5$.

Solution: $i)$ $32\pi(8 - 3\sqrt{3})/3$; $ii)$ $2\pi/3$; $iii)$ 8π; $iv)$ $493\pi/750$.

Problem 4. Compute the volume of the region limited by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.
Consider also the particular case $a = b = c = r$.

Solution: $i)$ $4\pi abc/3$; $ii)$ $4\pi r^3/3$.

Problem 5. Consider the region S in the plane defined by the curves mentioned below. Compute the mass and center of mass of S assuming that the density is constant:

i) $y = x^2$, $x + y = 2$,

ii) $y + 3 = x^2$, $x^2 = 5 - y$,

iii) $y = \sin^2 x,\ y = 0,\ x \in [0, \pi]$,

iv) $y = \sin x,\ y = \cos x,\ x \in [0, \pi/4]$.

Solution: $i)\ M = 9\rho/2;\ CM = (-1/2, 8/5);\ ii)\ M = 64\rho/3;\ CM = (0, 1);\ iii)\ M = \pi\rho/2;\ CM = (\pi/2, 3/8);\ iv)\ M = (\sqrt{2} - 1)\rho;\ CM = (\pi(2 + \sqrt{2})/4 - \sqrt{2} - 1, (\sqrt{2} + 1)/4)$.

Problem 6. Compute the mass for the plate corresponding to the region of the first quadrant of the circle $x^2 + y^2 \leq 4$, whose density is proportional to the distance to the centre of the circle.

Solution: $M = \frac{4k\pi}{3}$.

Problem 7. Let S be the region of the plane limited by the following curves:

i) $y = x^2, x + y = 2$;

ii) $y + 3 = x^2, x^2 = 5 - y$.

Compute the mass and the center of mass of S assuming that the density ρ is constant.

Problem 8. Compute the moment of inertia with respect to the vertical axis of the solid

$$V = \left\{x^2 + y^2 + z^2 \leq 4,\ z \geq \sqrt{x^2 + y^2}\right\}$$

(Assume a constant density ρ).

Problem 9. Compute the coordinates of the center of mass of the plate

$$M = \{(x, y) \in \mathbb{R}^2\,,\ 1 \leq x \leq 2,\ 1 \leq y \leq 3\}$$

where the density is given by the function $f(x, y) = xy$.

Solution: $(14/9, 13/6)$.

Problem 10. A metal plate is given by the the set of points in the plane

$$P = \{(x, y) \in \mathbb{R}^2\,,\ |y| \leq x \leq 1\}$$

with density $f(x, y) = y^2$. Compute the center of mass and the moments of inertia with respect to both axes.

Solution: $CM = (4/5, 0);\ \mathrm{I}_x = 1/15;\ \mathrm{I}_y = 1/9$.

Problem 11. i) Compute the area of the set $D = \{\,x = r\,\cos^3 t,\ y = r\,\sin^3 t,\ \ 0 \leq r \leq 1,\ 0 \leq t \leq \pi/2\} = \{\,x^{2/3} + y^{2/3} \leq 1,\ x, y \geq 0\}$.

ii) Compute the coordinates of the center of mass of D assuming constant density.

Solution: $i)$ $3\pi/32$; $ii)$ $x_{CM} = y_{CM} = 256/(315\pi)$.

Problem 12. The square Q of vertices $(0,0), (0,1), (1,0), (1,1)$ represents a plate of constant density ρ. Compute the moment of inertia around the line $x = y$.

Solution: $\mathrm{I}_E = \int_Q d((x,y),E)^2 \rho\, dx\, dy = \int_0^1 \int_0^1 \frac{(x-y)^2}{2} \rho\, dx\, dy = \frac{\rho}{12}$.

Problem 13. The temperature at the points of the cube $[-1,1]^3$ is proportional to the square of the distance to the origin.

i) Compute the mean temperature of the cube.

ii) At which points does the temperature coincide with the mean temperature?

Solution:

i) The average temperature in the cube will be

$$\frac{\int_W T(x,y,z)\, dx\, dy\, dz}{\int_W \, dx\, dy\, dz} = k.$$

ii) The points of the cube where the temperature coincides with the average are those points that satisfy

$$k(x^2 + y^2 + z^2) = k \Rightarrow x^2 + y^2 + z^2 = 1,$$

so that they will be the points of the sphere centered at the origin and of radius 1.

Problem 14. Compute the center of mass of a semispherical solid of radius R where the density at a point is given by the square of the distance between this point and the origin.

Solution: $(0, 0, 5R/12)$.

Problem 15. An ice cream consists of a cone with angle α, and a semisphere of radius R. The cone and the ball have constant densities, ρ_c and ρ_h, respectively. Find the ratio ρ_c/ρ_h for which the center of mass of the ice-cream is on the plane separating the cone from the ball.

Solution: $3\tan^2\alpha$.